

SPECTRAL SYNTHESIS FOR OPERATOR SPACE PROJECTIVE TENSOR PRODUCT OF C^* -ALGEBRAS

RANJANA JAIN AND AJAY KUMAR

ABSTRACT. We study the spectral synthesis for the Banach $*$ -algebra $A\widehat{\otimes}B$, the operator space projective tensor product of C^* -algebras A and B . It is shown that if A or B has finitely many closed ideals, then $A\widehat{\otimes}B$ obeys spectral synthesis. The Banach algebra $A\widehat{\otimes}A$ with the reverse involution is also studied.

1. INTRODUCTION AND NOTATIONS

For operator spaces V and W , and $u \in V \otimes W$, the *operator space projective tensor norm* is defined as

$$\|u\|_{\wedge} = \inf\{\|\alpha\|\|v\|\|w\|\|\beta\| : u = \alpha(v \otimes w)\beta\},$$

where $\alpha \in \mathbb{M}_{1,pq}$, $\beta \in \mathbb{M}_{pq,1}$, $v \in M_p(V)$ and $w \in M_q(W)$, $p, q \in \mathbb{N}$ being arbitrary, and $v \otimes w = (v_{ij} \otimes w_{kl})_{(i,k),(j,l)} \in M_{pq}(V \otimes W)$. The *operator space projective tensor product* $V\widehat{\otimes}W$ is the completion of $V \otimes W$ under $\|\cdot\|_{\wedge}$ -norm. The algebraic tensor product $V \otimes W$ is complete with respect to $\|\cdot\|_{\wedge}$ -norm if and only if either V or W is finite dimensional. To see this, if V is finite dimensional, then as a Banach space it is isomorphic to \mathbb{C}^n , thus $V\widehat{\otimes}W$ is Banach space isomorphic to the direct sum of n -copies of W , which is complete. Also, if V and W are both infinite dimensional, then one can choose two sequences $\{e_n\}$ and $\{f_n\}$ of linearly independent vectors in V and W such that $\|e_n\| = \|f_n\| = 1$ for all $n \in \mathbb{N}$. The sequence (u_n) in $V \otimes W$ defined as $u_n = \sum_{i=1}^n 2^{-i} e_i \otimes f_i$ is a Cauchy sequence with respect to $\|\cdot\|_{\wedge}$ -norm, but is not convergent in $V \otimes W$. It is known that for C^* -algebras A and B , $A\widehat{\otimes}B$ is a Banach $*$ -algebra under natural involution ([14]).

The notion of spectral synthesis has been studied extensively for commutative and unital Banach algebras, for L^1 -group algebras and for Banach $*$ -algebras [20, 6, 7, 13]. Spectral synthesis for Banach space projective tensor product of commutative Banach algebras and for the Haagerup tensor product of C^* -algebras has also been explored([13, 8, 1, 7]). Roughly speaking spectral synthesis holds for a Banach $*$ -algebra X if every closed ideal of X is the intersection of primitive ideals containing it. Spectral synthesis for Banach space projective tensor product of commutative Banach algebras has already been explored([13]). For commutative C^* -algebras A and B , the natural contractive homomorphism of $A\widehat{\otimes}B$ into $A \otimes^h B$ is an isomorphism whose inverse has norm

2000 *Mathematics Subject Classification.* 46L06, 46L07, 47L25, 43A45.

Key words and phrases. C^* -algebras, operator space projective tensor norm, spectral synthesis, hull-kernel topology.

equal to Grothendieck constant. Thus, for countable locally compact Hausdorff spaces X and Y , $C_0(X) \widehat{\otimes} C_0(Y)$ has spectral synthesis. However, for cantor set or any infinite compact group D , $C(D) \widehat{\otimes} C(D)$ does not have spectral synthesis ([8, 11.2.1], [13]).

In Section 2, we define the concept of spectral ideals in $A \widehat{\otimes} B$, and prove that the Banach $*$ -algebra $A \widehat{\otimes} B$ has spectral synthesis if and only if each closed ideal of $A \widehat{\otimes} B$ is spectral. This result is then used to produce plenty of spectral ideals in $A \widehat{\otimes} B$. We also discuss few cases where $A \widehat{\otimes} B$ obeys spectral synthesis. In particular, we prove that if A or B has finitely many closed ideals, then $A \widehat{\otimes} B$ has spectral synthesis. Thus, the Banach $*$ -algebras like $C_0(X) \widehat{\otimes} \mathcal{B}(H)$, $\mathcal{B}(H) \widehat{\otimes} \mathcal{K}(H)$ and $\mathcal{B}(H) \widehat{\otimes} \mathcal{B}(H)$ all obey spectral synthesis, X being a locally compact topological space and H being an infinite dimensional separable Hilbert space. In section 3, the algebra $A \widehat{\otimes} B$ with the reverse involution is discussed. It is shown that with this involution the algebra is symmetric and $*$ -semisimple only in the trivial cases.

For a Banach algebra X , we denote the set of closed (two-sided) ideals of X by $Id(X)$, the set of proper closed ideals of X by $Id'(X)$ and the set of all prime ideals by $Prime(X)$. If X is a Banach $*$ -algebra, then $Prim(X)$ stands for the set of primitive ideals of X , that is, the set of all kernels of irreducible $*$ -representations of X on Hilbert space. There is a topology τ_w on $Id(X)$ which is generated by the sub-basic open sets of the form

$$Z_J := \{I \in Id(X) : I \not\supseteq J\}, J \in Id(X).$$

We throughout use the notation q_J for the quotient map $q_J : A \rightarrow A/J$. Recall that, for closed ideals M and N of C^* -algebras A and B , the map $q_M \otimes q_N : A \otimes B \rightarrow A/M \otimes B/N$ extends to quotient maps $q_M \widehat{\otimes} q_N : A \widehat{\otimes} B \rightarrow A/M \widehat{\otimes} B/N$ and $q_M \otimes^{\min} q_N : A \otimes^{\min} B \rightarrow A/M \otimes^{\min} B/N$.

Let A and B be C^* -algebras. Define a map $\Phi : Id(A) \times Id(B) \rightarrow Id(A \widehat{\otimes} B)$ as

$$\Phi(M, N) = A \widehat{\otimes} N + M \widehat{\otimes} B.$$

The map Φ is well defined by [12, Proposition 3.2]. It satisfies many nice topological properties listed as below:

PROPOSITION 1.1. *Let A and B be C^* -algebras and $\Phi : Id(A) \times Id(B) \rightarrow Id(A \widehat{\otimes} B)$ be defined as above. Then*

- (i) Φ maps $Prime(A) \times Prime(B)$ onto $Prime(A \widehat{\otimes} B)$.
- (ii) Φ maps $Prim(A) \times Prim(B)$ into $Prim(A \widehat{\otimes} B)$. If A and B are separable, then Φ maps $Prim(A) \times Prim(B)$ onto $Prim(A \widehat{\otimes} B)$.
- (iii) Φ maps $Id'(A) \times Id'(B)$ into $Id'(A \widehat{\otimes} B)$ injectively.
- (iv) The mapping Φ is τ_w -continuous.
- (v) The restriction of Φ to $Id'(A) \times Id'(B)$ is a homeomorphism onto its image in $Id'(A \widehat{\otimes} B)$.
- (vi) The restriction of Φ to $Prime(A) \times Prime(B)$ is a homeomorphism onto $Prime(A \widehat{\otimes} B)$.

PROOF: (i) and (ii) follow from Theorems 3.1 and 3.2 of [11], respectively.

For (iii), note that, for proper closed ideals M and N of A and B , the isomorphism of $A/M \widehat{\otimes} B/N$ onto $(A \widehat{\otimes} B)/(A \widehat{\otimes} N + M \widehat{\otimes} B)$ ([11, Lemma 2.2])

assures that $A \widehat{\otimes} N + M \widehat{\otimes} B$ is also proper in $A \widehat{\otimes} B$. Further, for $M_1, M_2 \in Id'(A), N_1, N_2 \in Id'(B)$, $A \widehat{\otimes} N_1 + M_1 \widehat{\otimes} B \subseteq A \widehat{\otimes} N_2 + M_2 \widehat{\otimes} B$ if and only if $M_1 \subseteq M_2, N_1 \subseteq N_2$. To see this, consider an $m \in M_1$, so that for an arbitrary $b \in B$, $m \otimes b \in \ker(q_{M_1} \widehat{\otimes} q_{N_1}) \subseteq \ker(q_{M_2} \widehat{\otimes} q_{N_2})$, giving $q_{M_2}(m) = 0$, that is $m \in M_2$, and similarly $N_1 \subseteq N_2$. Thus, Φ is injective.

(iv)-(vi) can be proved exactly on the same lines of their counterparts in Haagerup tensor product as discussed in Lemma 1.4 and Theorem 1.5 of [2]. \square

Throughout this paper A and B represent C^* -algebras, until otherwise specified.

2. SPECTRAL SYNTHESIS

We first give the standard definition of spectral synthesis for a Banach $*$ -algebra that appear in the literature. Let X be a Banach $*$ -algebra. For each $E \subseteq Prim(X)$, there associates a closed ideal *kernel* of E defined as

$$k(E) := \cap_{P \in E} P.$$

Also, for each $M \subseteq X$, *hull* of M is defined as

$$h_X(M) := \{P \in Prim(X) : P \supseteq M\}.$$

We shall denote the hull of M by $h(M)$, when there is no confusion with X . Equip $Prim(X)$ with the *hull-kernel topology* (or, hk-topology), where for every $E \subseteq Prim(X)$, its closure is $\overline{E} = h(k(E))$. Similarly, one can talk about the hk-topology on $Prime(X)$. Note that, if $E \subseteq Prime(X)$, then the relative τ_w -topology on E coincides with the hull-kernel topology.

DEFINITION 2.1. A closed subset E of $Prim(X)$ is called *spectral* if $k(E)$ is the only closed ideal in X with hull equal to E . A Banach $*$ -algebra X is said to have *spectral synthesis* if every closed subset of $Prim(X)$ is spectral.

A closed ideal of Banach $*$ -algebra X is said to be semisimple if it is an intersection of all the primitive ideals of X containing it.

PROPOSITION 2.2. *Let X be a Banach $*$ -algebra having Wiener property. Then X has spectral synthesis if and only if for every $J \in Id(X)$, $J = k(h(J))$, or, in other words, every closed ideal of X is semisimple.*

PROOF: Let us consider a proper closed ideal J of X . Since X has Wiener property, there exists an irreducible $*$ -representation, say π , of X which annihilates J , that is, $J \subseteq \ker \pi$, so that $E = h(J)$ is non empty. We claim that E is closed in the hk-topology. Let $Q \in \overline{E} = h(k(E))$, then $k(E) \subseteq Q$. Since $J \subseteq P$ for all $P \in E$ we have $J \subseteq k(E) \subseteq Q$, so that $Q \in E$. which gives that E is closed. Since X obeys spectral synthesis, and $E = h(J)$, we have $J = k(E)$, that is, J is the intersection of primitive ideals containing it. Also, note that since X has Wiener property, the empty set ϕ is spectral, so that $X = k(h(X))$.

Converse follows easily from the given condition. \square

COROLLARY 2.3. *Let X be a Banach $*$ -algebra having Wiener property. Then X has spectral synthesis if and only if there is a one-one correspondence between the closed ideals of X and the τ_w -open subsets of $Prim(X)$ (or, $Prime(X)$).*

PROOF: Let X have spectral synthesis. For $J \in Id(X)$, recall $Z_J := \{P \in Prim(X) : P \not\supseteq J\} = Prim(X) \setminus h(J)$ is an open subset of $Prim(X)$ under the relative τ_w -topology, so that we have a well defined correspondence $J \mapsto Z_J$ between the closed ideals of X and τ_w -open subsets of $Prim(X)$. For $K, L \in Id(X)$, it is clear from Proposition 2.2 that $K = k(h(K))$, and $L = k(h(L))$. Thus, it can be easily seen that

$$K \subseteq L \quad \text{if and only if} \quad Z_K \subseteq Z_L,$$

which shows that the correspondence is one-one. Now consider a τ_w -open subset G of $Prim(X)$, and set $J := k(Prim(X) \setminus G)$. Since $Prim(X) \setminus G$ is closed under the hull-kernel topology,

$$Z_J = Prim(X) \setminus h(k(Prim(X) \setminus G)) = Prim(X) \setminus (Prim(X) \setminus G) = G,$$

which proves that this correspondence is surjective.

Conversely, for every closed ideal I of X , since $h(I) = h(k(h(I)))$, we have $Z_I = Z_{k(h(I))}$. Using the given condition, this gives $I = k(h(I))$. Result now follows from Proposition 2.2. \square

REMARK 2.4. For C^* -algebras A and B , since $A \hat{\otimes} B$ has Wiener property ([12, Theorem 4.1]), $A \hat{\otimes} B$ has spectral synthesis if and only if every closed ideal J of $A \hat{\otimes} B$ is semisimple. In particular, if $A \hat{\otimes} B$ has spectral synthesis then every closed ideal J of $A \hat{\otimes} B$ is the intersection of prime ideals containing J .

The next two results connect the spectral synthesis of a Banach $*$ -algebra with that of its ideal and the corresponding quotient algebra. The first result follows on the similar lines as that in [7, Proposition 1.16]. However, we present here a proof for the sake of completion.

PROPOSITION 2.5. *Let X be a Banach $*$ -algebra with Wiener property, and J be a closed $*$ -ideal of X having bounded approximate identity and Wiener property. If J and X/J both have spectral synthesis (as Banach $*$ -algebras), then X has spectral synthesis.*

PROOF: By Corollary 2.3, it is sufficient to show that for $I, K \in Id(X)$, $I = K$, whenever $h_X(I) = h_X(K)$. Note that, since X has Wiener property, X/J also has Wiener property, so by Proposition 2.2, every closed ideal of J and X/J is semisimple. For $P \in h_{X/J}(q_J(I))$, $P = \ker \pi$ with $\pi(q_J(I)) = \{0\}$, $\pi : X/J \rightarrow \mathcal{B}(H)$ being an irreducible $*$ -representation. Then $\pi_0 := \pi \circ q_J$ is an irreducible $*$ -representation of X on H with $\pi_0(I) = 0$. Since $h_X(I) = h_X(K)$, $\pi_0 \in h_X(K)$, which further gives $P \in h_{X/J}(q_J(K))$. Thus, $h_{X/J}(q_J(I)) = h_{X/J}(q_J(K))$. Since J has an approximate identity, by [5, Proposition 2.4], $I + J$ and $K + J$ are closed in X , so that $q_J(I)$ and $q_J(K)$ are closed ideals of X/J . Since X/J obeys spectral synthesis, by Proposition 2.2, $q_J(I) = q_J(K)$. Further, for any closed ideal L of J , it is routine to check that there is a one-one correspondence between the sets $\{P \in h_X(L) : J \not\subseteq P\}$ and $h_J(L)$ via $P \mapsto P \cap J$. Also, $h_X(I \cap J) = h_X(I) \cup h_X(J) = h_X(K) \cup h_X(J) = h_X(K \cap J)$. Thus, it can be easily seen that $h_J(I \cap J) = h_J(K \cap J)$. Since J has spectral synthesis, this gives, $I \cap J = K \cap J$.

Now, consider $x \in I$, then $q_J(x) = q_J(y)$ for some $y \in K$, so that $a := x - y \in J$. Let J_a be the smallest closed ideal of J containing a . Since J obeys spectral synthesis, $J_a = \cap\{P \in \text{Prim}(J) : J_a \subseteq P\}$. Clearly $\overline{JaJ} \subseteq J_a$. Now consider $P \in \text{Prim}(J)$ such that $JaJ \subseteq P$. Since P is prime being primitive, this gives $a \in P$ which shows that $J_a \subseteq P$. Thus $\overline{JaJ} = \cap\{P \in \text{Prim}(J) : JaJ \subseteq P\} = \cap\{P \in \text{Prim}(J) : J_a \subseteq P\} = J_a$. So

$$\begin{aligned} x - y \in \overline{J(x - y)J} &\subseteq \overline{JxJ - JyJ} \\ &\subseteq \overline{JIJ - JKJ} \\ &\subseteq \overline{I \cap J - K \cap J} \\ &= K \cap J. \end{aligned}$$

So, $x = y - (y - x) \in K + (K \cap J) = K$, which gives $I \subseteq K$. Similarly, $K \subseteq I$, which proves the claim. \square

In fact, the converse of the above statement is also true as presented below.

PROPOSITION 2.6. *Let X be a Banach $*$ -algebra with a closed $*$ -ideal J such that X and J both possess Wiener property. If X obeys spectral synthesis, then so does J and X/J .*

PROOF: By Proposition 2.2, it is enough to check that for a closed ideal L of J , $L = k(h_J(L))$. Since every closed ideal of X is semisimple, and every primitive ideal is prime, from [7, Proposition 1.14], L is also a closed ideal of X , so that by Proposition 2.2, $L = k(h_X(L))$. It can be easily verified that there is a one-one correspondence between the sets $\{P \in h_X(L) : J \not\subseteq P\}$ and $h_J(L)$ via $P \mapsto P \cap J$. So, we have

$$\begin{aligned} L = L \cap J &= \bigcap_{P \in h_X(L)} (P \cap J) \\ &= \left(\bigcap_{\substack{P \in h_X(L) \\ J \not\subseteq P}} (P \cap J) \right) \cap \left(\bigcap_{\substack{P \in h_X(L) \\ J \subseteq P}} (P \cap J) \right) \\ &= \left(\bigcap_{P' \in h_J(L)} P' \right) \cap J \\ &= k(h_J(L)) \end{aligned}$$

Thus, J obeys spectral synthesis.

Next, consider a closed ideal K of X/J . Since X/J has Wiener property, it is enough to check that $K \supseteq k(h_{X/J}(K))$. Consider an element $x \in k(h_{X/J}(K))$, where $x = y + J \in X/J$. Note that $K = I/J$ for some closed ideal I of X containing J . Using the one-one correspondence between $\text{Prim}(X/J)$ and $\{P \in \text{Prim}(X) : J \subseteq P\}$, one can check that $y \in k(h_X(I))$. Since X has spectral synthesis, $I = k(h_X(I))$, so that $y \in I$, which shows that $x \in K$. Hence the result. \square

We are now prepared to discuss spectral synthesis for operator space projective tensor product $A \hat{\otimes} B$ of C^* -algebras A and B . Allen, Sinclair and Smith, in [1], defined the concept of spectral synthesis for the Haagerup tensor product

of C^* -algebras in a somewhat different flavor. In the same spirit, using the terminologies of [1], we give another definition for the spectral synthesis of $A \widehat{\otimes} B$. It is known that for any C^* -algebras A and B , the canonical $*$ -homomorphism $i : A \widehat{\otimes} B \rightarrow A \otimes^{\min} B$ is injective ([10, Corollary 1]), so that we can regard $A \widehat{\otimes} B$ as a $*$ -subalgebra of $A \otimes^{\min} B$. Consider a closed ideal J of $A \widehat{\otimes} B$ and let J_{\min} be the closure of $i(J)$ in $A \otimes^{\min} B$, in other words, J_{\min} is the min-closure of J in $A \otimes^{\min} B$. Now we associate two closed ideals, namely the upper and the lower ideals, with J as:

$$\begin{aligned} J_l &= \text{closure of span of all elementary tensors of } J \text{ in } A \widehat{\otimes} B, \\ J^u &= J_{\min} \cap (A \widehat{\otimes} B). \end{aligned}$$

Clearly $J_l \subseteq J \subseteq J^u$ for any closed ideal J of $A \widehat{\otimes} B$.

DEFINITION 2.7. A closed ideal J of $A \widehat{\otimes} B$ is said to be *spectral* if $J_l = J = J^u$.

The main aim of this section is to show that $A \widehat{\otimes} B$ has spectral synthesis if and only if its every closed ideal is spectral. We first characterize the upper ideals in terms of primitive ideals.

LEMMA 2.8. For closed ideals M and N of A and B ,

$$\ker(q_M \widehat{\otimes} q_N) = \ker(q_M \otimes^{\min} q_N) \cap A \widehat{\otimes} B.$$

PROOF: For $z \in A \widehat{\otimes} B$, let $\{z_n\}$ be a sequence in $A \otimes B$ such that $\lim_n \|z_n - z\|_{\wedge} = 0$, then

$$\|(q_M \widehat{\otimes} q_N)(z_n) - (q_M \widehat{\otimes} q_N)(z)\|_{\min} \leq \|q_M \widehat{\otimes} q_N\| \|z_n - z\|_{\wedge},$$

which shows that the sequence $\{(q_M \widehat{\otimes} q_N)(z_n)\}$ is convergent to $(q_M \widehat{\otimes} q_N)(z)$ in $A \otimes^{\min} B$. Also, $\|z_n - z\|_{\min} \leq \|z_n - z\|_{\wedge}$, so that $\lim_n \|z_n - z\|_{\min} = 0$, which further gives

$$(q_M \otimes^{\min} q_N)(z_n) \xrightarrow{\min} (q_M \otimes^{\min} q_N)(z).$$

Since both the mappings $q_M \widehat{\otimes} q_N$ and $q_M \otimes^{\min} q_N$ agree on $A \otimes B$, by continuity, we have $(q_M \otimes^{\min} q_N)(z) = (q_M \widehat{\otimes} q_N)(z)$, and this is true for all $z \in A \widehat{\otimes} B$, proving the given relation. \square

PROPOSITION 2.9. For a closed ideal J of $A \widehat{\otimes} B$, $J = J^u$ if and only if J is *semisimple*.

PROOF: Let us first assume that $J = J^u$. Since, in a C^* -algebra every closed ideal is semisimple, $J_{\min} = \bigcap \{\ker \tilde{\pi}_{\alpha} : J_{\min} \subseteq \ker \tilde{\pi}_{\alpha}\}$, where each $\tilde{\pi}_{\alpha}$ is an irreducible $*$ -representation of $A \otimes^{\min} B$ on some Hilbert space. Set $\pi_{\alpha} := \tilde{\pi}_{\alpha} \circ i$, then each π_{α} is an irreducible $*$ -representation of $A \widehat{\otimes} B$ annihilating J . Using some routine calculations, and the fact that $J = J^u$ one can prove that $J = \bigcap \ker \pi_{\alpha}$. Note that, although the collection $\{P \in \text{Prim}(A \widehat{\otimes} B) : J \subseteq P\}$ is larger than $\{\ker \pi_{\alpha} : \pi_{\alpha} = \tilde{\pi}_{\alpha} \circ i\}$, it is easy to check that J is actually the intersection of all the primitive ideals of $A \widehat{\otimes} B$ containing J .

Conversely, let $J = \bigcap_{J \subseteq P_{\alpha}} P_{\alpha}$, P_{α} being primitive ideals of $A \widehat{\otimes} B$. Let, if possible, there exist an element $x \in J^u$ such that $x \notin J$. Then $x \notin P_{\alpha}$ for some α . Since P_{α} is primitive, by [11, Theorem 3.2], there exist closed (prime) ideals M and N in A and B , respectively, such that $P_{\alpha} = A \widehat{\otimes} N + M \widehat{\otimes} B$.

Now, consider the bounded homomorphisms $q_M \widehat{\otimes} q_N : A \widehat{\otimes} B \rightarrow A/M \widehat{\otimes} B/N$, and $q_M \otimes^{\min} q_N : A \otimes^{\min} B \rightarrow A/M \otimes^{\min} B/N$ with $\ker(q_M \widehat{\otimes} q_N) = P_\alpha$ ([12, Proposition 3.5]. By Lemma 2.8, $x \notin \ker(q_M \otimes^{\min} q_N)$, which by Hahn Banach Theorem gives a $\phi \in (A \otimes^{\min} B)^*$ such that $\phi(x) \neq 0$, and $\phi(\ker(q_M \otimes^{\min} q_N)) = \{0\}$. The relation $J \subseteq P_\alpha \subseteq \ker(q_M \otimes^{\min} q_N)$ gives $J_{\min} \subseteq \ker(q_M \otimes^{\min} q_N)$, which further shows that $\phi(J_{\min}) = 0$. Thus $x \notin J_{\min}$, which gives a contradiction to the fact that $x \in J^u$. Hence the result. \square

Using Propositions 2.2 and 2.9, we have a following characterization for spectral synthesis in terms of upper ideals.

THEOREM 2.10. *The Banach $*$ -algebra $A \widehat{\otimes} B$ has spectral synthesis if and only if $J = J^u$, for every closed ideal J of $A \widehat{\otimes} B$.*

We now prove that the Banach $*$ -algebra $A \widehat{\otimes} B$ has spectral synthesis if and only if every closed ideal of $A \widehat{\otimes} B$ is spectral. We borrow some ideas from [15] to prove the same. We first need an elementary result.

LEMMA 2.11. *Let J_i and K_i be closed ideals of C^* -algebras A_i , $i = 1, 2$. Then $J_1 \widehat{\otimes} J_2 \subseteq A_1 \widehat{\otimes} K_2 + K_1 \widehat{\otimes} A_2$ if and only if either $J_1 \subseteq K_1$ or $J_2 \subseteq K_2$.*

THEOREM 2.12. *For C^* -algebras A and B , the Banach $*$ -algebra $A \widehat{\otimes} B$ has spectral synthesis if and only if every closed ideal of $A \widehat{\otimes} B$ is spectral.*

PROOF: We just need to prove that for every closed ideal J of $A \widehat{\otimes} B$, $J = J_l$, if $A \widehat{\otimes} B$ has spectral synthesis. Using Corollary 2.3, it is sufficient to show that $Z_J \subseteq Z_{J_l}$, where $Z_J := \{P \in \text{Prime}(A \widehat{\otimes} B) : P \not\supseteq J\}$. Set $X := \text{Prime}(A \widehat{\otimes} B)$ and consider an element P of Z_J . Since Z_J is an open subset of X and $\Phi : \text{Prime}(A) \times \text{Prime}(B) \rightarrow X$ is continuous, there exist open subsets U_1, U_2 of $\text{Prime}(A)$ and $\text{Prime}(B)$ such that $\Phi(U_1 \times U_2) \subseteq Z_J$ and $P \in \Phi(U_1 \times U_2)$. Let $J_1 \in \text{Id}(A)$, $J_2 \in \text{Id}(B)$ be the corresponding closed ideals such that $U_i = Z_{J_i}$, $i = 1, 2$. We claim that $Z_{J_1 \widehat{\otimes} J_2} = \Phi(U_1 \times U_2) = \Phi(Z_{J_1} \times Z_{J_2})$. For any $Q \in Z_{J_1 \widehat{\otimes} J_2}$, by definition, $J_1 \widehat{\otimes} J_2 \not\subseteq Q$. Since $Q \in X$, and Φ is onto (Proposition 1.1), there exists $Q_1 \in \text{Prime}(A)$, $Q_2 \in \text{Prime}(B)$ such that $A \widehat{\otimes} Q_2 + Q_1 \widehat{\otimes} B = \Phi(Q_1, Q_2) = Q$. By Lemma 2.11, $J_1 \not\subseteq Q_1$ and $J_2 \not\subseteq Q_2$. This implies that $Q_i \in Z_{J_i} = U_i$, so that $Q = \Phi(Q_1, Q_2) \in \Phi(U_1 \times U_2)$. Thus, $Z_{J_1 \widehat{\otimes} J_2} \subseteq \Phi(U_1 \times U_2)$. For the other containment, consider $\Phi(K_1, K_2) \in \Phi(Z_{J_1} \times Z_{J_2})$. Since $K_i \in Z_{J_i}$, we have $J_i \not\subseteq K_i$, so by Lemma 2.11, $J_1 \widehat{\otimes} J_2 \not\subseteq \Phi(K_1, K_2)$. Note that $\Phi(K_1, K_2) \in X$, thus by the definition, $\Phi(K_1, K_2) \in Z_{J_1 \widehat{\otimes} J_2}$. So, $Z_{J_1 \widehat{\otimes} J_2} \subseteq Z_J$, which further gives, $J_1 \widehat{\otimes} J_2 \subseteq J$. But, the definition of J_l says that $J_1 \widehat{\otimes} J_2 \subseteq J_l$. This means that $Z_{J_1 \widehat{\otimes} J_2} \subseteq Z_{J_l}$. Since, $P \in \Phi(U_1 \times U_2) = Z_{J_1 \widehat{\otimes} J_2}$, this gives $P \in Z_{J_l}$. Thus $Z_J \subseteq Z_{J_l}$, which proves that $J \subseteq J_l$, and hence the result. \square

REMARK 2.13. In other words, if $A \widehat{\otimes} B$ obeys spectral synthesis, then every closed ideal J of $A \widehat{\otimes} B$ is the closure of the sum of all product ideals $J_1 \widehat{\otimes} J_2 \subseteq J$, where $J_1 \in \text{Id}(A)$, $J_2 \in \text{Id}(B)$.

The Banach $*$ -algebra $A \widehat{\otimes} B$ contains plenty of spectral ideals as demonstrated in the following and some later examples.

PROPOSITION 2.14. *For $I \in Id(A)$ and $J \in Id(B)$, the closed ideal $A \widehat{\otimes} J + I \widehat{\otimes} B$ of $A \widehat{\otimes} B$ is spectral. In particular, every closed maximal ideal, primitive ideal and prime ideal of $A \widehat{\otimes} B$ is spectral.*

PROOF: Set $K := A \widehat{\otimes} J + I \widehat{\otimes} B = \ker(q_I \widehat{\otimes} q_J)$, then it is clear from the definition that $K = K_l$. Consider an element $u \in K^u$. Let, if possible, $u \notin K$, then by Lemma 2.8, $u \notin \ker(q_I \otimes^{\min} q_J)$. Now, $K \subseteq \ker(q_I \otimes^{\min} q_J)$ implies $K_{\min} \subseteq \ker(q_I \otimes^{\min} q_J)$, giving $u \notin K_{\min}$, a contradiction. Thus K is spectral. Rest follows from the fact that every maximal, primitive and prime ideal can be expressed as an ideal of this form ([12, Theorem 3.10], [11, Theorem 3.1, 3.2]). \square

Next, we prepare the ingredients to prove that for an infinite dimensional separable Hilbert space H , $\mathcal{B}(H) \widehat{\otimes} \mathcal{B}(H)$ obeys spectral synthesis. We first need some elementary results regarding the lower and upper ideals of a closed ideal.

PROPOSITION 2.15. *For closed ideals J and K in $A \widehat{\otimes} B$, we have:*

- (a) $J_l \subseteq K_l$ and $J^u \subseteq K^u$, if $J \subseteq K$;
- (b) $(JK)_l = J_l K_l = J_l \cap K_l = (J \cap K)_l$, if J_l or K_l has a bounded approximate identity;
- (c) $(J \cap K)^u \subseteq J^u \cap K^u$, with equality if $J = J^u, K = K^u$.

PROOF: (a) is trivial. For (b), we first show that $J_l \cap K_l \subseteq J_l K_l$. Let $x \in J_l \cap K_l$ and assume that J_l has bounded approximate identity. By Cohen's Factorization Theorem, there exist $y, z \in J_l$ such that $x = yz$ and z belongs to the closed left ideal generated by x in J_l . Clearly, $z \in J_l \cap K_l$, so that $x \in J_l K_l$. Thus, $J_l \cap K_l \subseteq J_l K_l$. Now, for an elementary tensor x in $J \cap K$, clearly $x \in J_l \cap K_l$, giving $(J \cap K)_l \subseteq J_l \cap K_l$. Also, for $a = \sum_{i=1}^n x_i \otimes y_i \in J_l$ and $b = \sum_{j=1}^m z_j \otimes w_j \in K_l$, clearly $ab \in (JK)_l$, being an elementary tensor of JK . Since J_l and K_l are both generated by elementary tensors, routine calculations show that $J_l K_l \subseteq (JK)_l$. Thus, we have

$$(J \cap K)_l \subseteq J_l \cap K_l \subseteq J_l K_l \subseteq (JK)_l \subseteq (J \cap K)_l,$$

which gives the required equality. For (c), using the fact that $(J \cap K)_{\min} \subseteq J_{\min} \cap K_{\min}$, we get

$$(J \cap K)^u \subseteq J_{\min} \cap K_{\min} \cap A \widehat{\otimes} B = J^u \cap K^u.$$

\square

Following are some direct consequences of the above proposition.

COROLLARY 2.16. *If I and J are closed ideals of $A \widehat{\otimes} B$ with at least one of them having bounded approximate identity. Then $I \cap J$ is spectral, whenever I and J are spectral.*

COROLLARY 2.17. *Every product ideal of $A \widehat{\otimes} B$ is spectral. In particular, for closed ideals I and J of A and B , $I \widehat{\otimes} J = (I \otimes^{\min} J) \cap A \widehat{\otimes} B$.*

PROOF: For a product ideal $I \widehat{\otimes} J$ of $A \widehat{\otimes} B$, using [11, Proposition 2.4], we can write

$$I \widehat{\otimes} J = (A \widehat{\otimes} J) \cap (I \widehat{\otimes} B).$$

Also, from [12, Lemma 3.1], $A \widehat{\otimes} J$ and $I \widehat{\otimes} B$ both possess bounded approximate identities. Thus, from Proposition 2.14 and Corollary 2.16, $I \widehat{\otimes} J$ is spectral. Clearly,

$$I \widehat{\otimes} J = (I \widehat{\otimes} J)^u = (I \widehat{\otimes} J)_{\min} \cap A \widehat{\otimes} B = (I \otimes^{\min} J) \cap A \widehat{\otimes} B.$$

□

COROLLARY 2.18. *If either A or B is a simple C^* -algebra, then $A \widehat{\otimes} B$ obeys spectral synthesis.*

PROOF: Let A be simple. By [12, Theorem 3.8], every closed ideal of $A \widehat{\otimes} B$ is a product ideal and thus is spectral by Corollary 2.17. Using Theorem 2.12, we get $A \widehat{\otimes} B$ obeys spectral synthesis. □

In particular, for any C^* -algebra A , the Banach $*$ -algebras $A \widehat{\otimes} C_r^*(\mathbb{F}_2)$, $A \widehat{\otimes} A_\infty$ and $A \widehat{\otimes} \mathcal{K}(H)$ obey spectral synthesis, where $C_r^*(\mathbb{F}_2)$ is the C^* -algebra associated to the left regular representations of the free group \mathbb{F}_2 on two generators, A_∞ is the Glimm algebra ([18]) and $\mathcal{K}(H)$ is the C^* -algebra of compact operators on an infinite dimensional separable Hilbert space H .

THEOREM 2.19. *For an infinite dimensional separable Hilbert space H , the Banach $*$ -algebra $\mathcal{B}(H) \widehat{\otimes} \mathcal{B}(H)$ obeys spectral synthesis.*

PROOF: From [12, Theorem 3.11], we know that the only non trivial closed ideals of $\mathcal{B}(H) \widehat{\otimes} \mathcal{B}(H)$ are $\mathcal{K}(H) \widehat{\otimes} \mathcal{K}(H)$, $\mathcal{B}(H) \widehat{\otimes} \mathcal{K}(H)$, $\mathcal{K}(H) \widehat{\otimes} \mathcal{B}(H)$ and $\mathcal{B}(H) \widehat{\otimes} \mathcal{K}(H) + \mathcal{K}(H) \widehat{\otimes} \mathcal{B}(H)$. Using Proposition 2.14 and Corollary 2.17, we can see that all the proper closed ideals of $\mathcal{B}(H) \widehat{\otimes} \mathcal{B}(H)$ are spectral. The result now follows from Theorem 2.12. □

PROPOSITION 2.20. *Let A and B be C^* -algebras such that A or B has finitely many closed ideals. Then $A \widehat{\otimes} B$ obeys spectral synthesis.*

PROOF: Without loss of generality, we may assume that B has finitely many closed ideals say n , where $n \geq 2$. We prove the result by induction on n . For $n = 2$, B is simple and the result follows from Corollary 2.18. Let the result be true for all C^* -algebras with at most $(n - 1)$ ideals. Let B have $n > 2$ closed ideals. Since there are finitely many closed ideals of B , there exists a minimal (non-trivial) closed ideal, say K , of B , which is clearly simple. Consider the closed $*$ -ideal $J := A \widehat{\otimes} K$ of $X := A \widehat{\otimes} B$. Since K is simple, using Corollary 2.18, it is clear that J has spectral synthesis. Note that, by [11, Lemma 2.2(1)], X/J is isomorphic to $A \widehat{\otimes} (B/K)$ and the latter has spectral synthesis by induction hypothesis, since B/K has at most $(n - 1)$ closed ideal. So, X/J also has spectral synthesis. Moreover, J and X/J both have Wiener property ([11, Theorem 4.1]), and J has bounded approximate identity ([12, Lemma 3.1]), the result now follows from Proposition 2.5. □

Thus, for any C^* -algebra A , $A \widehat{\otimes} \mathcal{B}(H)$ obey spectral synthesis, where H is a separable infinite dimensional Hilbert space. In particular, $C_0(X) \widehat{\otimes} \mathcal{B}(H)$, $\mathcal{B}(H) \widehat{\otimes} \mathcal{B}(H)$ and $\mathcal{B}(H) \widehat{\otimes} \mathcal{K}(H)$ obey spectral synthesis, where X is a locally compact Hausdorff space. For more examples of C^* -algebras with finitely many closed ideals, see [16].

COROLLARY 2.21. *If A and B both have finite number of closed ideals, then every closed ideal of $A \widehat{\otimes} B$ is a finite sum of product ideals.*

PROOF: It follows from Proposition 2.20 and Remark 2.13. \square

REMARK 2.22. Let A and B be C^* -algebras. If A is subhomogeneous, then by [3, Proposition IV.1.4.6], every subhomogeneous C^* -algebra is bidual type I, so that A^{**} is a type I von Neumann algebra, which is then nuclear by Corollary IV.2.2.10 and Theorem IV.3.1.5 of [3]. Consider the Gelfand-Naimark semi norm on $A \otimes B$ defined as

$$\gamma(x) = \sup\{\|T(x)\|\},$$

where the supremum runs over all the $*$ -representations T of $A \otimes B$ on Hilbert spaces. Since the $\|\cdot\|_\gamma$ -norm on $A \otimes B$ is continuous with respect to ' \wedge '-norm, it can be extended to $A \widehat{\otimes} B$, and thus $A \otimes B$ is $\|\cdot\|_\gamma$ -dense in $(A \widehat{\otimes} B, \|\cdot\|_\gamma)$. By [17, Proposition 10.5.20], $A \widehat{\otimes} B$ is $*$ -regular, whenever $A \otimes B$ is so. Since A is nuclear, by [9, Corollary 2.7], $A \widehat{\otimes} B$ is $*$ -regular. It is also Hermitian ([11, Theorem 4.6] and is $*$ -semisimple (follows from [11, Theorem 4.1]). Hence, the definition of spectral synthesis in [6] is equivalent to our definition in this case.

In the case of commutative separable C^* -algebras the ideals which are not singly generated fail to be spectral. A similar result also holds true in the non-commutative situation. The following can be proved exactly on the same lines of [1, Theorem 6.12].

PROPOSITION 2.23. *Let A and B be separable C^* -algebras, and J be a non-zero closed ideal of $A \widehat{\otimes} B$. Then J is singly generated if it is spectral.*

3. REVERSE INVOLUTION

Let A be a C^* -algebra. On the Banach algebra $A \otimes A$ (with usual multiplication), define the involution as $(a \otimes b)^* = b^* \otimes a^*$ for all $a, b \in A$. Then it extends to an isometric involution on $A \widehat{\otimes} A$ and $A \widehat{\otimes} A$ forms a Banach $*$ -algebra with this involution, which we denote by $A \widehat{\otimes}_r A$. Regarding the closed $*$ -ideals of $A \widehat{\otimes}_r A$, note that the closed ideals of $A \widehat{\otimes}_r A$ coincide with the ones in $A \widehat{\otimes} A$; however, the closed $*$ -ideals differ. We do not know whether a closed ideal of $A \widehat{\otimes} A$ is a $*$ -ideal or not, but in $A \widehat{\otimes}_r A$ a closed ideal need not be a $*$ -ideal. For example, in the space $\mathcal{B}(H) \widehat{\otimes}_r \mathcal{B}(H)$, the closed ideals $\mathcal{K}(H) \widehat{\otimes} \mathcal{B}(H)$ and $\mathcal{B}(H) \widehat{\otimes} \mathcal{K}(H)$ are not $*$ -ideals. In fact, it has only two non-trivial closed $*$ -ideals, namely $\mathcal{K}(H) \widehat{\otimes}_r \mathcal{K}(H)$ and $\mathcal{B}(H) \widehat{\otimes} \mathcal{K}(H) + \mathcal{K}(H) \widehat{\otimes} \mathcal{B}(H)$.

We know that with natural involution $A \widehat{\otimes} A$ has a faithful $*$ -representation and is always $*$ -semisimple for any C^* -algebra A . However, we show that this is not the case with $A \widehat{\otimes}_r A$.

PROPOSITION 3.1. *Let A be a unital C^* -algebra. Then, $A \widehat{\otimes}_r A$ has a faithful $*$ -representation if and only if $A = \mathbb{C}I$, I being the unity of A .*

PROOF: Let π be a faithful $*$ -representation of $A \widehat{\otimes}_r A$ on a Hilbert space H . Define $\pi_1(a) := \pi(1 \otimes a)$ and $\pi_2(a) := \pi(a \otimes 1)$ for all $a \in A$. Then π_1 and π_2

are both bounded representations of A on $\mathcal{B}(H)$, with $\pi(a \otimes b) = \pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a)$ for all $a, b \in A$. Also

$$(1) \quad \pi_1(a^*) = \pi(1 \otimes a^*) = \pi((a \otimes 1)^*) = (\pi(a \otimes 1))^* = \pi_2(a)^*$$

for all $a \in A$. It is known that an element $h \in A$ is self adjoint if and only if $\|\exp it h\| = 1$ for all $t \in \mathbb{R}$. For a self adjoint element $h \in A$, using the facts that π is contractive and that $\|\cdot\|_\wedge$ -norm is a cross norm, we have

$$\begin{aligned} \|\exp it \pi_1(h)\| &= \|\pi(\exp it(h \otimes 1))\| \\ &\leq \|\exp it(h \otimes 1)\|_\wedge \\ &= \lim_m \left\| \sum_{n=1}^m \frac{i^n t^n (h^n \otimes 1)}{n!} \right\|_\wedge \\ &= \lim_m \left\| \left(\sum_{n=1}^m \frac{i^n t^n h^n}{n!} \right) \otimes 1 \right\|_\wedge \\ &= \lim_m \left\| \sum_{n=1}^m \frac{i^n t^n h^n}{n!} \right\| \\ &= \|\exp it h\| = 1, \end{aligned}$$

and this is true for all $t \in \mathbb{R}$. Thus, $\|\exp it \pi_1(h)\| = 1$ for all $t \in \mathbb{R}$, which shows that $\pi_1(h)$ is a self adjoint element of $\mathcal{B}(H)$. This, combined with equation (1), gives $\pi_1(h) = \pi_2(h)$, that is $\pi(1 \otimes h) = \pi(h \otimes 1)$. Since π is faithful, $1 \otimes h = h \otimes 1$. So, for any $\phi \in A^*$, $\phi(1)h = \phi(h)1$, which further gives $h \in \mathbb{C}I$, and this is true for any self adjoint element h of A . Since any $a \in A$ can be written as $a = h + ik$, h and k being self adjoint elements of A , we obtain the required result. \square

COROLLARY 3.2. (i) $A \widehat{\otimes}_r A$ is $*$ -semisimple if and only if $A = \mathbb{C}I$.
(ii) $A \widehat{\otimes}_r A$ is symmetric if and only if $A = \mathbb{C}I$

PROOF: (i) Follows easily from the fact that a semisimple Banach $*$ -algebra possesses a faithful $*$ -representation [19, Corollary 4.7.16].

(ii) Let $A \widehat{\otimes}_r A$ be symmetric. Using the same argument as in [1, Proposition 5.16], one can show that the radical of $A \widehat{\otimes}_r A$ is $\{0\}$. By [19, Theorem 4.7.15], $*$ -radical of $A \widehat{\otimes}_r A$ coincides with its radical. Thus $A \widehat{\otimes}_r A$ is $*$ -semisimple, which using above part implies $A = \mathbb{C}I$. \square

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DEPARTMENT OF MATHEMATICS, LADY SHRI RAM COLLEGE FOR WOMEN, NEW DELHI-110024, INDIA.

E-mail address: ranjanaj_81@rediffmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI, DELHI-110007, INDIA.

E-mail address: akumar@maths.du.ac.in